

Every monomorphism of the Lie algebra of unitriangular polynomial derivations is an automorphism

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Abstract

We prove that every monomorphism of the Lie algebra \mathfrak{u}_n of unitriangular derivations of the polynomial algebra $P_n = K[x_1, \dots, x_n]$ is an automorphism.

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1 Introduction

Throughout, K is a field of characteristic zero and K^* is its group of units; $P_n := K[x_1, \dots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K -linear derivations) of P_n ; $\text{Aut}_K(P_n)$ is the group of automorphisms of the polynomial algebra P_n ; $\text{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of K -derivations of P_n ; $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} Kx^\alpha \partial^\beta$ is the n 'th Weyl algebra; for each natural number $n \geq 2$,

$$\mathfrak{u}_n := K\partial_1 + P_1\partial_2 + \cdots + P_{n-1}\partial_n$$

is the Lie algebra of unitriangular polynomial derivations (it is a Lie subalgebra of the Lie algebra $\text{Der}_K(P_n)$) and $G_n := \text{Aut}_K(\mathfrak{u}_n)$ is its group of automorphisms; $\delta_1 := \text{ad}(\partial_1), \dots, \delta_n := \text{ad}(\partial_n)$ are the inner derivations of the Lie algebra \mathfrak{u}_n determined by the elements $\partial_1, \dots, \partial_n$ (where $\text{ad}(a)(b) := [a, b]$).

The aim of the paper is to prove the following theorem.

Theorem 1.1 *Every monomorphism of the Lie algebra \mathfrak{u}_n is an automorphism.*

Remark. Not every epimorphism of the Lie algebra \mathfrak{u}_n is an automorphism. Moreover, there are countably many distinct ideals $\{I_{i\omega^{n-1}} \mid i \geq 0\}$ such that

$$I_0 = \{0\} \subset I_{\omega^{n-1}} \subset I_{2\omega^{n-1}} \subset \cdots \subset I_{i\omega^{n-1}} \subset \cdots$$

and the Lie algebras $\mathfrak{u}_n/I_{i\omega^{n-1}}$ and \mathfrak{u}_n are isomorphic (Theorem 5.1.(1), [4]).

Theorem 1.1 has bearing of the Conjecture of Dixmier [7] for the Weyl algebra A_n over a field of characteristic zero that claims: *every homomorphism of the Weyl algebra is an automorphism*. The Weyl algebra A_n is a simple algebra, so every algebra endomorphism of A_n is a monomorphism. This conjecture is open since 1968 for all $n \geq 1$. It is stably equivalent to the Jacobian Conjecture for the polynomial algebras as was shown by Tsuchimoto [9], Belov-Kanel and Kontsevich [6], (see also [2] for a short proof). The Jacobian Conjecture claims that *certain* monomorphisms of the polynomial algebra P_n are isomorphisms: *Every algebra endomorphism σ of the polynomial algebra P_n such that $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ is an automorphism*. The condition that $\mathcal{J}(\sigma) \in K^*$ implies that the endomorphism σ is a monomorphism.

An analogue of the Conjecture of Dixmier is true for the algebra $\mathbb{I}_1 := K\langle x, \frac{d}{dx}, f \rangle$ of polynomial integro-differential operators.

Theorem 1.2 (Theorem 1.1, [3]) *Each algebra endomorphism of \mathbb{I}_1 is an automorphism.*

In contrast to the Weyl algebra $A_1 = K\langle x, \frac{d}{dx} \rangle$, the algebra of polynomial differential operators, the algebra \mathbb{I}_1 is neither a left/right Noetherian algebra nor a simple algebra. The left localizations, $A_{1,\partial}$ and $\mathbb{I}_{1,\partial}$, of the algebras A_1 and \mathbb{I}_1 at the powers of the element $\partial = \frac{d}{dx}$ are isomorphic. For the simple algebra $A_{1,\partial} \simeq \mathbb{I}_{1,\partial}$, there are algebra endomorphisms that are not automorphisms [3].

Before giving the proof of Theorem 1.1, let us recall several results that are used in the proof.

The derived series for the Lie algebra \mathfrak{u}_n . Let \mathcal{G} be a Lie algebra over the field K and $\mathfrak{a}, \mathfrak{b}$ be its ideals. The *commutant* $[\mathfrak{a}, \mathfrak{b}]$ of the ideals \mathfrak{a} and \mathfrak{b} is the linear span in \mathcal{G} of all the elements $[a, b]$ where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Let $\mathcal{G}_{(0)} := \mathcal{G}$, $\mathcal{G}_{(1)} := [\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}_{(i)} := [\mathcal{G}_{(i-1)}, \mathcal{G}_{(i-1)}]$ for $i \geq 2$. The descending series of ideals of the Lie algebra \mathcal{G} ,

$$\mathcal{G}_{(0)} = \mathcal{G} \supseteq \mathcal{G}_{(1)} \supseteq \cdots \supseteq \mathcal{G}_{(i)} \supseteq \mathcal{G}_{(i+1)} \supseteq \cdots$$

is called the *derived series* for the Lie algebra \mathcal{G} . The Lie algebra \mathfrak{u}_n admits the finite strictly descending chain of ideals

$$\mathfrak{u}_{n,1} := \mathfrak{u}_n \supset \mathfrak{u}_{n,2} \supset \cdots \supset \mathfrak{u}_{n,i} \supset \cdots \supset \mathfrak{u}_{n,n} \supset \mathfrak{u}_{n,n+1} := 0 \quad (1)$$

where $\mathfrak{u}_{n,i} := \sum_{j=i}^n P_{j-1} \partial_j$ for $i = 1, \dots, n$. For all $i < j$,

$$[\mathfrak{u}_{n,i}, \mathfrak{u}_{n,j}] \subseteq \begin{cases} \mathfrak{u}_{n,i+1} & \text{if } i = j, \\ \mathfrak{u}_{n,j} & \text{if } i < j. \end{cases} \quad (2)$$

(Proposition 2.1.(2), [4]) states that (1) is the derived series for the Lie algebra \mathfrak{u}_n , i.e., $(\mathfrak{u}_n)_{(i)} = \mathfrak{u}_{n,i+1}$ for all $i \geq 0$.

The group of automorphisms of the Lie algebra \mathfrak{u}_n . In [5], the group of automorphisms G_n of the Lie algebra \mathfrak{u}_n of unitriangular polynomial derivations is found ($n \geq 2$), it is isomorphic to an iterated semi-direct product (Theorem 5.3, [5]),

$$\mathbb{T}^n \ltimes (\text{UAut}_K(P_n)_n \ltimes (\mathbb{F}'_n \times \mathbb{E}_n))$$

where \mathbb{T}^n is an algebraic n -dimensional torus, $\text{UAut}_K(P_n)_n$ is an explicit factor group of the group $\text{UAut}_K(P_n)$ of unitriangular polynomial automorphisms, \mathbb{F}'_n and \mathbb{E}_n are explicit groups that are isomorphic respectively to the groups \mathbb{I} and \mathbb{J}^{n-2} where $\mathbb{I} := (1 + t^2 K[[t]], \cdot) \simeq K^{\mathbb{N}}$ and $\mathbb{J} := (tK[[t]], +) \simeq K^{\mathbb{N}}$. It is shown that the *adjoint group* of automorphisms $\mathcal{A}(\mathfrak{u}_n)$ of the Lie algebra \mathfrak{u}_n is equal to the group $\text{UAut}_K(P_n)_n$ (Theorem 7.1, [5]). Recall that the *adjoint group* $\mathcal{A}(\mathcal{G})$ of a Lie algebra \mathcal{G} is generated by the elements $e^{\text{ad}(g)} := \sum_{i \geq 0} \frac{\text{ad}(g)^i}{i!} \in \text{Aut}_K(\mathcal{G})$ where g runs through all the locally nilpotent elements of the Lie algebra \mathcal{G} (an element g is a *locally nilpotent element* if the inner derivation $\text{ad}(g) := [g, \cdot]$ of the Lie algebra \mathcal{G} is a locally nilpotent derivation). The group G_n contains the semi-direct product $\mathbb{T}^n \ltimes \mathcal{T}_n$ where

$$\mathcal{T}_n := \{\sigma \in \text{Aut}_K(P_n) \mid \sigma(x_1) = x_1, \sigma(x_i) = x_i + a_i \text{ where } a_i \in (x_1, \dots, x_{i-1}), i = 2, \dots, n\}$$

where (x_1, \dots, x_{i-1}) is the maximal ideal of the polynomial algebra $P_{i-1} := K[x_1, \dots, x_{i-1}]$ generated by the elements x_1, \dots, x_{i-1} .

Proof of Theorem 1.1. Let $\varphi : \mathfrak{u}_n \rightarrow \mathfrak{u}_n$ be a monomorphism of the Lie algebra \mathfrak{u}_n . By (Proposition 2.1.(2), [4]), $(\mathfrak{u}_n)_{(i)} = \mathfrak{u}_{n,i+1}$ for all i . So, the descending chain of ideals (1) is the derived series for the Lie algebra \mathfrak{u}_n of length $l(\mathfrak{u}_n) = n$ (by definition, this is the number of nonzero terms in the derived series). Clearly, $l(\mathfrak{u}_{n,2}) = n - 1$ and

$$l(\varphi(\mathfrak{u}_n)) = l(\mathfrak{u}_n) = n$$

$(\varphi(\mathfrak{u}_n) \simeq \mathfrak{u}_n)$. It follows that

$$\varphi(\mathfrak{u}_n) \not\subseteq \mathfrak{u}_{n,2}$$

since otherwise we would have $n = l(\varphi(\mathbf{u}_n)) \leq l(\mathbf{u}_{n,2}) = n - 1$, a contradiction. This means that $\partial'_1 := \varphi(\partial_1) = \lambda_1 \partial_1 + u_1$ for some $\lambda_1 \in K^*$ and $u_1 \in \mathbf{u}_{n,2}$. We use induction on i to show that

$$\partial'_i := \varphi(\partial_i) = \lambda_i \partial_i + u_i, \quad i = 1, \dots, n, \quad (3)$$

for some elements $\lambda_i \in K^*$ and $u_i \in \mathbf{u}_{n,i+1}$. In particular, $\partial_n = \lambda_n \partial_n$. The initial step, $i = 1$, has already been established. Suppose that $i \geq 2$ and that (3) holds for all numbers $i' < i$. Since $\varphi((\mathbf{u}_n)_{(j)}) \subseteq (\mathbf{u}_n)_{(j)}$ for all $j \geq 1$, we have the inclusion $\varphi(\mathbf{u}_{n,i}) = \varphi((\mathbf{u}_n)_{(i-1)}) \subseteq (\mathbf{u}_n)_{(i-1)} = \mathbf{u}_{n,i}$ which implies that $\partial'_i = \lambda_i \partial_i + u_i$ for some elements $\lambda_i \in P_{i-1}$ and $u_i \in \mathbf{u}_{n,i+1}$. It remains to show that $\lambda_i \in K^*$. This fact follows from the commutation relations $[\partial'_j, \partial'_i] = 0$ for $j = 1, \dots, i-1$ ($0 = \varphi([\partial_j, \partial_i]) = [\partial'_j, \partial'_i]$). In more detail, for $j = i-1$,

$$0 = [\partial'_{i-1}, \partial'_i] = [\lambda_{i-1} \partial_{i-1} + u_{i-1}, \lambda_i \partial_i + u_i] = \lambda_{i-1} \partial_{i-1} (\lambda_i) \partial_i + v_{i-1}$$

for some element $v_{i-1} \in \mathbf{u}_{n,i+1}$. Therefore, $\partial_{i-1}(\lambda_i) = 0$, i.e., $\lambda_i \in P_{i-2}$. Now, we use a second downward induction on j starting on $j = i-1$ to show that

$$\lambda_i \in P_j \quad \text{for all } j = 1, \dots, i-1. \quad (4)$$

The initial step, $j = i-1$, has been just proved. Suppose that (4) is true for all $j = k, \dots, i-1$. In particular, $\lambda_i \in P_k = K[x_1, \dots, x_k]$. We have to show that $\lambda_i \in P_{k-1}$. For, we use the equality $[\partial'_k, \partial'_i] = 0$:

$$0 = [\lambda_k \partial_k + u_k, \lambda_i \partial_i + u_i] = \lambda_k \partial_k (\lambda_i) \partial_i + v_k$$

for some element $v_k \in \mathbf{u}_{n,i+1}$ ($[u_k, \lambda_i \partial_i] \in \mathbf{u}_{n,i+1}$ since $\lambda_i \in P_k$ and $[\oplus_{k+1 \leq j \leq i} P_{j-1} \partial_j, \lambda_i \partial_i] = 0$). Therefore, $\partial_k(\lambda_i) = 0$, i.e., $\lambda_i \in P_{k-1}$. By induction on j , (4) holds. In particular, for $j = 1$: $\lambda_i \in P_{i-1} = P_0 = K$. We have to show that $\lambda_i \neq 0$. Notice that $\mathbf{u}_{n,i} = \oplus_{j=i}^n P_{j-1} \partial_j$, $l(\mathbf{u}_{n,i}) = n - i + 1$ and $\varphi(\mathbf{u}_{n,i}) \subseteq \mathbf{u}_{n,i}$. The monomorphism φ respects the Lie subalgebra $\mathcal{G} = K \partial_i + \mathbf{u}_{n,i+1}$ of the Lie algebra \mathbf{u}_n , i.e., $\varphi(\mathcal{G}) \subseteq \mathcal{G}$. The inclusion of Lie algebras $\mathcal{G} \subseteq \mathbf{u}_{n,i}$ yields the inequality $l(\mathcal{G}) \leq l(\mathbf{u}_{n,i}) = n - i + 1$ (the equality follows from the fact that $(\mathbf{u}_n)_{(j)} = \mathbf{u}_{n,j+1}$ for all $j \geq 0$). The vector space

$$\mathcal{H} = K \partial_i + K[x_i] \partial_{i+1} + K[x_i, x_{i+1}] \partial_{i+2} + \dots + K[x_i, \dots, x_{n-1}] \partial_n$$

is a Lie subalgebra of \mathcal{G} which is isomorphic to the Lie algebra \mathbf{u}_{n-i+1} . Therefore, $l(\mathcal{H}) = l(\mathbf{u}_{n-i+1}) = n - i + 1$. The inclusion of Lie algebra $\mathcal{H} \subseteq \mathcal{G}$ yields the inequality $n - i + 1 = l(\mathcal{H}) \leq l(\mathcal{G})$. Therefore, $l(\mathcal{G}) = n - i + 1$.

Suppose that $\lambda_i = 0$, we seek a contradiction. In that case, $\varphi(\mathcal{G}) \subseteq \mathbf{u}_{n,i+1}$ and so

$$n - i + 1 = l(\mathcal{G}) = l(\varphi(\mathcal{G})) \leq l(\mathbf{u}_{n,i+1}) = n - i,$$

a contradiction.

Therefore, (3) holds. By (Theorem 3.6.(2), [5]), there exists a unique automorphism $\sigma \in \mathbb{T}^n \ltimes \mathcal{T}_n \subseteq \mathcal{G}_n$ such that $\sigma(\partial_i) = \partial'_i$ for $i = 1, \dots, n$. By replacing the monomorphism φ by the monomorphism $\sigma^{-1} \varphi$, without loss of generality we can assume that

$$\partial'_i = \partial_i \quad \text{for all } i = 1, \dots, n.$$

The vector space \mathbf{u}_n is the union $\cup_{i \geq 0} N_i$ of vector subspaces $N_i := \{u \in \mathbf{u}_n \mid \delta_j^{i+1}(u) = 0, j = 1, \dots, n-1\}$ where $\delta_j = \text{ad}(\partial_j)$. Clearly, $N_i = \oplus_{j=1}^n N_i \cap P_{j-1} \partial_j$ and $N_i \cap P_{j-1} \partial_j = \oplus \{Kx^\alpha \mid \alpha = (\alpha_1, \dots, \alpha_{j-1}) \in \mathbb{N}^{j-1}, \alpha_k \leq i \text{ for } k = 1, \dots, j-1\}$. In particular,

$$\dim_K(N_i) < \infty \quad \text{for all } i \geq 0.$$

By the very definition of the vector spaces N_i and the fact that $\varphi(\partial_i) = \partial_i$ for $i = 1, \dots, n-1$,

$$\varphi(N_i) \subseteq N_i \quad \text{for all } i \geq 0.$$

Since the linear map φ is an injection and the vector spaces N_i are finite dimensional, we have $\varphi(N_i) = N_i$ for all $i \geq 0$, i.e., φ is a bijection. \square

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References

- [1] V. V. Bavula, The inversion formula for automorphisms of the Weyl algebras and polynomial algebras, *J. Pure Appl. Algebra* **210** (2007), 147-159; arXiv:math.RA/0512215.
- [2] V. V. Bavula, The Jacobian Conjecture_{2n} implies the Dixmier Problem_n, ArXiv:math.RA/0512250.
- [3] V. V. Bavula, An analogue of the Conjecture of Dixmier is true for the algebra of polynomial integro-differential operators, Arxiv:math.RA: 1011.3009.
- [4] V. V. Bavula, Lie algebras of unitriangular polynomial derivations and an isomorphism criterion for their Lie factor algebras, Arxiv:math.RA/1204.4908.
- [5] V. V. Bavula, The groups of automorphisms of the Lie algebras of unitriangular polynomial derivations, Arxiv:math.AG/1204.4910.
- [6] A. Belov-Kanel and M. Kontsevich, The Jacobian conjecture is stably equivalent to the Dixmier Conjecture, *Mosc. Math. J.* **7** (2007), no. 2, 209–218 (arXiv:math. RA/0512171).
- [7] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.
- [8] B. Rotman and G. T. Kneebone, The theory of sets and transfinite numbers. Oldbourne, London, 1966, 144 pp.
- [9] Y. Tsuchimoto, Endomorphisms of Weyl algebra and p -curvatures. *Osaka J. Math.* **42** (2005), no. 2, 435–452.

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